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Noncrystallographic Coxeter group H_4 in E_8

Mehmet Koca[1], **Ramazan Koç**[2] **and Muataz Al-Barwani**[1]

[1] Department of Physics, College of Science, Sultan Qaboos University, PO Box 36, Al-Khod 123, Muscat, Sultanate of Oman

[2] Department of Physics, Gaziantep University, 27310 Gaziantep, Turkey

E-mail: kocam@squ.edu.om, koc@gul1.bim.gantep.edu.tr and muataz@squ.edu.om

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Abstract

The E_8 lattice is constructed in terms of icosians by matching two sets of F_4 lattices described by quaternions. Embedding the noncrystallographic group H_4 into the Weyl group $W(E_8)$ has been described using matrix generators with an emphasis on the relevant Coxeter elements. The conjugacy classes of H_4 in terms of quaternions and the characters of the two four-dimensional irreducible representations are explicitly calculated.

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1. Introduction

Noncrystallographic icosahedral structures in three-dimensional space can be understood by embedding the point symmetry group $2 \times A_5 \approx H_3$ into the crystallographic point group $W(D_6)$ [1]. The noncrystallographic Coxeter group H_3 can be generated by three reflections leading to a root system described by 30 pure unit quaternions [2]. A profound mathematical structure normally arises in the description of the noncrystallographic Coxeter group H_4 in the four-dimensional space where the root system of H_4 is represented by 120 unit icosians, quaternionic elements of the binary icosahedral group $2A_5$ [3]. A powerful method has been invented by Wilson [4] where the set of unit icosians and their σ multiples ($\sigma = (1-\sqrt{5})/2$) constitute the root system of E_8 [5]. In an earlier paper [6] one of us (MK) indicated that the E_8 root system described by icosians can also be obtained by matching two sets of F_4 roots represented by quaternions. Further properties of embedding H_4 in $W(E_8)$ have been described in a number of papers [7].

Recent developments in superstring theories, particularly in the heterotic $E_8 \times E_8$ superstring theory [8] motivate further studies of the E_8 lattice and its symmetries. In section 2 we give a brief summary of what has been achieved for H_4 and E_8 lattices in relation to icosians. In section 3 we discuss the matrix representations of the Weyl group $W(E_8)$ generators on the icosian basis. The generators of H_4 as a subgroup of $W(E_8)$ are transformed into the block-diagonal forms where two four-dimensional irreducible representations of H_4 become manifest. Section 4 deals with the Coxeter element of $W(E_8)$ and its relevance to the Coxeter element of H_4 where its characteristic polynomial can be factored into two polynomials, one of which is the characteristic polynomial of H_4. We discuss H_4 as the largest finite subgroup of $O(4)$ [9] in section 5. Class structures of H_4 and the determination of the number of elements in each conjugacy class have been worked out in section 6 where characters of the representations of concern are also tabulated. Finally, in the conclusion we remark on the method we employed and on its possible use in physics. In the appendix we list the generators of $W(E_8)$ in the basis of quaternion units and their multiples by σ.

2. Quaternionic root systems, magic square and icosians

A quaternion $q = \sum_{a=0}^{3} q_a e_a$ with q_a real numbers and $e_a(e_0 = 1, e_1, e_2, e_3)$ quaternion units is a vector in four-dimensional Euclidean space where pure quaternions satisfy the relations

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \qquad \overline{e_i} = -e_i \qquad i, j, k = 1, 2, 3. \tag{1}$$

Here δ_{ij} and ϵ_{ijk} are the usual Kronecker and Levi-Civita symbols respectively. The quaternions of unit norm $q\overline{q} = \overline{q}q = 1$ with $\overline{q} = q_0 - \sum_{i=1}^{3} q_i e_i$ form a group isomorphic to $SU(2)$. The finite subgroups of quaternions, also known as the binary polyhedral groups [10] are the cyclic groups $\langle n, n, 1\rangle$ of order $2n$, the dicyclic groups $\langle n, 2, 2\rangle$ of order $4n$, the binary tetrahedral group $\langle 3, 3, 2\rangle$ of order 24, the binary octahedral group $\langle 4, 3, 2\rangle$ of order 48 and the binary icosahedral group $\langle 5, 3, 2\rangle$ of order 120. The quaternionic elements of the binary icosahedral group are called icosions. In the four-dimensional space the root systems of the crystallographic groups $W(D_4)$ and $W(F_4)$ and noncrystallographic Coxeter group H_4 can be described by quaternions. Under the quaternion scalar product

$$(q_1, q_2)_Q = \tfrac{1}{2}(\overline{q_1}q_2 + \overline{q_2}q_1) \tag{2}$$

the following set of quaternions describe a scaled root system of F_4 [6]:

A_0	A_1	A_2	A_2
$\pm 1, \pm e_1, \pm e_2, \pm e_3$	$\frac{1}{2}(\pm 1 \pm e_1)$	$\frac{1}{2}(\pm 1 \pm e_2)$	$\frac{1}{2}(\pm 1 \pm e_3)$
$\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$	$\frac{1}{2}(\pm e_2 \pm e_3)$	$\frac{1}{2}(\pm e_3 \pm e_1)$	$\frac{1}{2}(\pm e_1 \pm e_2)$.

A pairing of the two sets of quaternionic F_4 roots in the following form:

$$(0, A_0), (A_0, 0), (A_1, A_3), (A_3, A_2), (A_2, A_1) \tag{3}$$

where $(A_i, A_j) = A_i + \sigma A_j (\sigma = (1-\sqrt{5})/2, \tau = (1+\sqrt{5})/2)$ constitute the quaternionic roots of E_8 provided one introduces the Euclidean scalar product

$$(q_1, q_2)_Q \quad \Rightarrow \quad (q_1, q_2)_E \tag{4}$$

where σ and τ are replaced respectively by $\sigma \to 0$ and $\tau \to 1$ in the quaternion scalar product (2) [4]. Indeed half of the roots in (3) are the icosians q which constitute the root system of H_4 under the quaternion scalar product. The remaining half is of the form σq. Any pair of unit icosians q_1, q_2 satisfy the quaternion scalar product $(q_1, q_2)_Q = a$ where $a = 0, \pm\frac{1}{2}, \pm\frac{\tau}{2}, \pm\frac{\sigma}{2}$. Now under the Euclidean scalar product the same pair of quaternions satisfy $(q_1, q_2)_E = b$ where $b = 0, \pm\frac{1}{2}, \pm\frac{1}{2}, 0$ respectively. Similarly, Euclidean scalar products of the forms $(q_1, \sigma q_2)_E = (\sigma q_1, q_2)_E, (\sigma q_1, \sigma q_2)_E$ can take the values $0, \pm\frac{1}{2}$. The icosians constituting the root system of H_4 are classified in table 1 according to the conjugacy

Table 1. Icosians with respect to their conjugacy classes.

Conjugacy classes and orders of elements	Elements in the conjugacy classes denoted by their numbers (cyclic permutations in e_1, e_2, e_3 should be added to get the right number of elements in each class)
1	1
2	-1
10	$12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3)$,
5	$12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3)$,
10	$12'_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2)$,
5	$12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2)$,
6	$20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3)$, $\frac{1}{2}(1 \pm \tau e_1 \pm \sigma e_2)$,
3	$20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3)$, $\frac{1}{2}(-1 \pm \tau e_1 \pm \sigma e_2)$,
4	$30 \ : 15_+ : e_1, e_2, e_3, \frac{1}{2}(\sigma e_1 \pm \tau e_2 \pm e_3)$,
	$: 15_- : -e_1, -e_2, -e_3, \frac{1}{2}(-\sigma e_1 \pm \tau e_2 \pm e_3)$,

Table 2. The magic square of lattice matching.

	$SU(3)$	$SP(3)$	F_4
$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$SP(3)$	$SU(6)$	$SO(12)$	E_7
F_4	E_6	E_7	E_8

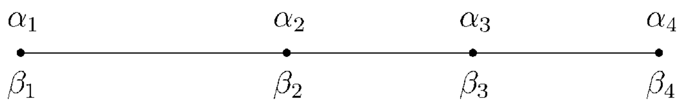

Figure 1. The Coxeter diagram of H_4.

classes of the binary icosahedral group $2A_5$. The numbers in front of the sets denote the numbers of elements in each conjugacy class; +(−1) signs indicate the sign of the first entity. Note that 30 pure quaternions are in the same conjugacy class of $2A_5$ and can be taken as the roots of the Coxeter graph H_3. The lattice matching (3) of the form (F_4, F_4) is a special case of the magic square [11] given in table 2.

Let us denote by $\alpha_1 = -e_1$, $\alpha_2 = (\tau e_1 + e_2 + \sigma e_3)/2$, $\alpha_3 = -e_2$ and $\alpha_4 = (\sigma + e_2 + \tau e_3)/2$, the simple roots of H_4. The Coxeter graph of H_4 is illustrated in figure 1 where $\beta_i\,(i = 1, 2, 3, 4)$ are the reflection generators of H_4. Then the Coxeter–Dynkin diagram of E_8 can be taken as shown in figure 2. By letting

$$\begin{array}{llll} l_1 - l_2 = -\sigma\alpha_4 & l_2 - l_3 = -\sigma\alpha_3 & l_3 - l_4 = -\sigma\alpha_2 & l_4 - l_5 = \alpha_1 \\ l_5 - l_6 = \alpha_2 & l_6 - l_7 = \alpha_3 & l_7 - l_0 = \alpha_4 & l_6 + l_7 = -\sigma\alpha_1 \end{array} \tag{5}$$

where $l_0 = (l_1 + \cdots + l_8)/2$ one can relate the simple roots of our choice to the set of orthogonal vectors $l_i (i = 1, \ldots, 8)$ normalized by $1/\sqrt{2}$. The generators β_i of H_4 are given by $\beta_i = r_i r'_i = r'_i r_i$ (no summation over i) [12].

Now we prove that the successive applications of r_i and r'_i, each of which requiring a Euclidean scalar product, leads to the quaternion scalar product for β_i. Let us note that the quaternion scalar product of icosians can be written in the form $(q_1, q_2)_Q = a + b\sigma$ where a and b are rational numbers, however $(q_1, q_2)_E = a$. Consider now the actions of r_i and r'_i on an arbitrary quaternion:

$$r_i : q \to q' = q - 2(\alpha_i, q)_E \alpha_i = q - 2a\alpha_i \tag{6}$$

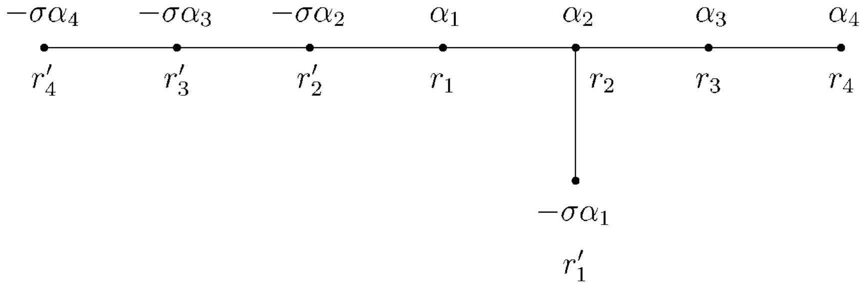

Figure 2. The Coxeter–Dynkin diagram of E_8.

where $(\alpha_i, q)_E = a$ when $(\alpha_i, q)_Q = a + b\sigma$.

Next, we apply r_i' on q'

$$r_i' : q' \to q'' = q' - 2(\alpha_i', q')_E \alpha_i' = q - 2a\alpha_i - 2(\alpha_i', q')_E \alpha_i'. \tag{7}$$

Since $\alpha_i' = -\sigma\alpha_i$ we obtain $(\alpha_i', q')_E = -b$ and

$$\begin{aligned} \beta_i = r_i r_i' :\quad & q \to q'' = q - 2(a + b\sigma)\alpha_i \\ & q \to q - 2(\alpha_i, q)_Q \alpha_i = -\alpha_i \overline{q} \alpha_i. \end{aligned} \tag{8}$$

This also shows that σq also transforms under β_i in the same manner as q:

$$\beta_i : \sigma q \to -\alpha_i \sigma \overline{q} \alpha_i. \tag{9}$$

The results in (8)–(9) prove that the roots of E_8 split under H_4 into two disjoint sets, icosians q and σq, which implies that the H_4 generators can be put into block-diagonal forms.

3. Matrix representations of the generators of H_4

One can choose $e_a (a = 0, 1, 2, 3)$ as the orthogonal basis for icosians of the H_4 root system. The σ multiples of these units σe_a extend the space to eight-dimensional Euclidean space when the Euclidean scalar product is invoked. In the appendix we list the matrix representations of the E_8 generators r_i and $r_i' (i = 1, 2, 3, 4)$ in the $e_a, \sigma e_a$ basis. Since we are interested only in its subgroup H_4 below we give the eight-dimensional reducible representation of the generators β_i in the basis $e_a, \sigma e_a$:

$$\beta_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & -1 & 0 & 0 & & 0 & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & 0 & & & 0 & -1 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 1 \end{array}\right] \beta_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\beta_3 = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & 0 & & \\ 0 & 0 & -1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & 0 & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & -1 & 0 \\ & & & & 0 & 0 & 0 & 1 \end{array}\right] \beta_4 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

These matrices can be transformed into block diagonal form by an orthogonal similarity transformation

$$\beta_i' = A\beta_i A^T \tag{11}$$

where $A = A^T = A^{-1}$ and is given by

$$\begin{aligned} A &= \left[\begin{array}{c|c} xI & yI \\ \hline yI & -xI \end{array}\right] \\ &= x\left[\begin{array}{c|c} I & \tau I \\ \hline \tau I & -I \end{array}\right]. \end{aligned} \tag{12}$$

Here I is a 4×4 unit matrix, $x = \sqrt{(2+\sigma)/5} = 0.5257\ldots$ and $y = \sqrt{(2+\tau)/5} = 0.8506\ldots$ with some properties $y = \tau x$, $x = -\sigma y$, $x^2 + y^2 = 1$, $y^2 + 2xy = \tau$, $x^2 - 2xy = \sigma$.

The 8×8 reducible generators of H_4 are now in block diagonal form:

$$\begin{aligned} \beta_1' &= \beta_1 \\ \beta_3' &= \beta_3 \\ \beta_2' &= \frac{1}{2}\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & & & & \\ 0 & \tau & -\sigma & 1 & & 0 & & \\ 0 & -\sigma & 1 & -\tau & & & & \\ 0 & 1 & -\tau & \sigma & & & & \\ \hline & & & & 2 & 0 & 0 & 0 \\ & 0 & & & 0 & \sigma & -\tau & 1 \\ & & & & 0 & -\tau & 1 & -\sigma \\ & & & & 0 & 1 & -\sigma & \tau \end{array}\right] \\ \beta_4' &= \frac{1}{2}\left[\begin{array}{cccc|cccc} \sigma & 0 & -\tau & 1 & & & & \\ 0 & 2 & 0 & 0 & & 0 & & \\ -\tau & 0 & 1 & -\sigma & & & & \\ 1 & 0 & -\sigma & \tau & & & & \\ \hline & & & & \tau & 0 & -\sigma & 1 \\ & 0 & & & 0 & 2 & 0 & 0 \\ & & & & -\sigma & 0 & 1 & -\tau \\ & & & & 1 & 0 & -\tau & \sigma \end{array}\right]. \end{aligned} \tag{13}$$

They act on the new basis $\eta_a' = xe_a + y\sigma e_a$ and $\eta_a = ye_a - x\sigma e_a$ $(a = 0, 1, 2, 3)$ which can also be expressed in terms of the familiar vectors $l_i (i = 1, 2, \ldots, 8)$. The first four unit vectors η_a' are a basis for the upper block matrices and the η_a are the basis vectors for the lower block matrices. It is obvious from the matrices in (13) that the eight-dimensional defining representation of $W(E_8)$ branches as $\mathbf{8} = \mathbf{4} \oplus \mathbf{4}'$ where $\mathbf{4}$ and $\mathbf{4}'$ are the irreducible representations of H_4, which has four four-dimensional irreducible representations. Our notation for the irreducible representation of H_4 is explained in section 6.

4. Coxeter elements and Coxeter exponents

The matrix $M' = \beta'_1\beta'_2\beta'_3\beta'_4$ can be taken as the Coxeter element of E_8 which is already in block diagonal form

$$M' = \frac{1}{2}\left[\begin{array}{cccc|cccc} \sigma & 0 & -\tau & 1 & & & & \\ -1 & -\tau & 0 & \sigma & & & 0 & \\ 0 & -\sigma & -1 & -\tau & & & & \\ -\tau & 1 & -\sigma & 0 & & & & \\ \hline & & & & \tau & 0 & -\sigma & 1 \\ & & 0 & & -1 & -\sigma & 0 & \tau \\ & & & & 0 & -\tau & -1 & -\sigma \\ & & & & -\sigma & 1 & -\tau & 0 \end{array}\right]. \tag{14}$$

The characteristic equation of the Coxeter element M' can be written as

$$\left|M' - \lambda I\right| = \lambda^8 + \lambda^7 - \lambda^5 - \lambda^4 - \lambda^3 + \lambda^2 + 1 = p(\lambda)g(\lambda) = 0 \tag{15}$$

where $p(\lambda) = \lambda^4 + \tau\lambda^3 + \tau\lambda^2 + \tau\lambda + 1 = 0$ leads to the eigenvalues of the upper block matrix and $g(\lambda) = \lambda^4 + \sigma\lambda^3 + \sigma\lambda^2 + \sigma\lambda + 1 = 0$ leads to the eigenvalues of the lower block matrix. The solutions of $p(\lambda) = 0$ are the complex exponents of the form $\lambda = \exp(\mathrm{i}m(2\pi)/30)$ where m takes half the Coxeter exponents of $W(E_8)$, $m = 7, 13, 17, 23$ and the solutions of $g(\lambda) = 0$ can be expressed as the same exponent where m takes the other half of the Coxeter exponents of $W(E_8)$, $m = 1, 11, 19, 29$ [3, 10]. The last set of Coxeter exponents are also the Coxeter exponents of H_4 where the order of the group is $2 \cdot 12 \cdot 20 \cdot 30 = 14\,400$. Clearly, the H_4 is a subgroup of $W(E_8)$ with an index $8 \cdot 14 \cdot 18 \cdot 24 = 48\,384$, which upon multiplication by the order of H_4, 14 400, yields the order of $W(E_8)$, $192 \cdot 10!$.

The Coxeter element of the $D_6 \approx SO(12)$ subgroup of E_8 can be written as $N' = \beta'_1\beta'_2\beta'_3$ which is, in block diagonal form

$$N' = \frac{1}{2}\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & & & & \\ 0 & -\tau & -\sigma & -1 & & & 0 & \\ 0 & -\sigma & -1 & -\tau & & & & \\ 0 & 1 & \tau & \sigma & & & & \\ \hline & & & & 2 & 0 & 0 & 0 \\ & & 0 & & 0 & -\sigma & -\tau & -1 \\ & & & & 0 & -\tau & -1 & -\sigma \\ & & & & 0 & 1 & \sigma & \tau \end{array}\right] \tag{16}$$

thereby showing that η'_0 and η_0 are left invariant. This means N' can be taken as a 6×6 block-diagonal matrix acting on the space spanned by η'_i and $\eta_i (i = 1, 2, 3)$ which are linear combinations of pure quaternions e_i and the $\sigma e_i (i = 1, 2, 3)$. The characteristic equation of N' can be written as

$$\left|N' - \lambda I\right| = \lambda^6 + \lambda^5 + \lambda + 1 = h(\lambda)k(\lambda) = 0 \tag{17}$$

where

$$h(\lambda) = \lambda^3 + \tau\lambda^2 + \tau\lambda + 1 = 0$$

and

$$k(\lambda) = \lambda^3 + \sigma\lambda^2 + \sigma\lambda + 1 = 0.$$

The solutions of $h(\lambda) = k(\lambda) = 0$ are complex exponentials $\exp(\mathrm{i}m(2\pi)/10)$ where $m = 3, 5, 7$ for $h(\lambda) = 0$ and $m = 1, 5, 9$ for $k(\lambda) = 0$. Therefore the lower matrix is the

Coxeter element of the group $H_3 \approx 2 \times A_5$ of order $2 \cdot 6 \cdot 10 = 120$ with an index $9 \cdot 6 \cdot 8 = 192$ in D_6 the order of which is $2^5 \cdot 6!$. Further restriction to the product $K' = \beta_1' \beta_2'$ would lead to a block-diagonal 4×4 matrix which is the Coxeter element of the Weyl group of $A_4 \approx SU(5)$ of order 120. The lower 2×2 matrix is the Coxeter element of the noncrystallographic Coxeter group H_2 of order 10.

By this reduction we have shown how the sequence of embedding $A_4 \subset D_6 \subset E_8$ leads to the embedding of the corresponding noncrystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$ in their crystallographic groups.

One more remark would be informative before we end this section. Coxeter elements and the incidence matrices $C = A - 2I$ where A and I are the Cartan matrix and unit matrix respectively have the same eigenvalues [10]. Therefore one could obtain the same information from the incidence matrix of E_8.

5. H_4 as the largest finite subgroup of $O(4)$

The quaternion group is isomorphic to $SU(2)$ which is in turn $2 \to 1$ homomorphic to $SO(3)$. A pair of unit quaternions (p, r) multiplying a quaternion q from the left and right

$$(p, r) : q \to pqr \tag{18}$$

leaves the quaternion norm $q\overline{q} = \overline{q}q$ invariant. Therefore (p, r) is an element of $O(4)$, indeed an element of $SO(4)$. The transformation in (18) has a geometrical interpretation. Any quaternion p can be written

$$p = \cos\alpha + \boldsymbol{P}\sin\alpha = \exp(\alpha\boldsymbol{P}) \tag{19}$$

where $\boldsymbol{P}$ is a pure quaternion $\boldsymbol{P}^2 = -1$, $\overline{\boldsymbol{P}} = -\boldsymbol{P}$. One can prove that a general displacement

$$(p, r) : q \to \mathrm{e}^{\alpha\boldsymbol{P}} q \mathrm{e}^{\beta\boldsymbol{R}} \qquad \boldsymbol{R}^2 = -1 \qquad \overline{\boldsymbol{R}} = -\boldsymbol{R}$$

is a double rotation through angles $\alpha + \beta$ about the plane generated by the vectors 0, $\boldsymbol{P} - \boldsymbol{R}$, $1 + \boldsymbol{PR}$ and $\alpha - \beta$ about the plane generated by the vectors 0, $\boldsymbol{P} + \boldsymbol{R}$, $1 - \boldsymbol{PR}$ [3]. These two planes are obviously orthogonal to each other.

In addition to the transformation in (18) one can define a transformation

$$(p, r)^* : q \to p\overline{q}r \tag{20}$$

which also leaves $q\overline{q} = \overline{q}q$ invariant. Since (20) leaves $p + r$ invariant and changes the sign of $p - r$ for general unit quaternions p and r, it follows that $(p, r)^*$ is a rotary reflection.

The preceding arguments lead to the result that the transformation (p, r) can be represented by matrices of determinant +1 while $(p, r)^*$ corresponds to the transformations of determinant -1. Therefore the transformations (p, r) form a subgroup $SO(4)$. Below we give some properties of the elements of $O(4)$:

$$\begin{aligned} &(a, b)(c, d) = (ac, db) \\ &(a, b)(a, b)^{-1} = (1, 1) = (-1, -1) \Rightarrow (a, b)^{-1} = (a^{-1}, b^{-1}) = (\bar{a}, \bar{b}) \\ &(a, b)^*(c, d)^* = (a\bar{d}, \bar{c}b) \\ &(a, b)^{*^{-1}}(a, b)^* = (1, 1) = (-1, -1) \Rightarrow (a, b)^{*^{-1}} = (b, a)^* \\ &(a, b)(c, d)^* = (ac, db)^* \\ &(a, b)^*(c, d) = (a\bar{d}, \bar{c}b)^*. \end{aligned} \tag{21}$$

Clearly the centre of $O(4)$ is represented by the elements $(1, 1) = (-1, -1)$, $(-1, 1) = (1, -1)$ and $(1, 1)^* = (-1, -1)^*$, $(-1, 1)^* = (1, -1)^*$ which form the group $Z_2 \times Z_2$. We have the isomorphisms

$$\frac{O(4)}{Z_2 \times Z_2} \approx \frac{SO(4)}{Z_2} \approx SO(3) \times SO(3) \approx \frac{SU(2)}{Z_2} \times \frac{SU(2)}{Z_2}. \tag{22}$$

All the finite subgroups of $O(4)$ are classified by du Val [9]. The largest finite subgroup of $O(4)$ is, as expected, the noncrystallographic Coxeter group H_4.

Without referring to the matrix representation, it is tempting to prove that the Coxeter number h of the group H_4 is 30. To find $M = \beta_1\beta_2\beta_3\beta_4$ we successively apply the β_i to q yielding

$$M : q \to \alpha_1\overline{\alpha_2}\alpha_3\overline{\alpha_4}q\overline{\alpha_4}\alpha_3\overline{\alpha_2}\alpha_1 \qquad M = (\alpha_1\overline{\alpha_2}\alpha_3\overline{\alpha_4}, \overline{\alpha_4}\alpha_3\overline{\alpha_2}\alpha_1). \tag{23}$$

Here $\alpha_i (i = 1, 2, 3, 4)$ are the simple roots of H_4 given in section 2. Then M reads

$$M = (p, r) = \left(\tfrac{1}{2}(\sigma + \tau e_1 + e_3), -\tfrac{1}{2}(1 + e_1 - e_2 + e_3)\right) \tag{24}$$

where p and r belong to the conjugacy classes 12_+ and 20_-. Let us find h when

$$M^h = (1, 1) = (-1, -1) = (p^h, r^h). \tag{25}$$

We know from table 1 that $p^{10} = 1, r^3 = 1$ so that their least common multiple is $h = 30$.

6. Determination of conjugacy classes of the H_4 and characters of some representations

The character table of H_4 has been determined by Grove [13]. Our approach to determine the conjugacy classes is highly different and more explicit. We notice that the elements $(p, r) = (-p, -r)$ from a subgroup H_4' of order 7200 which is a discrete subgroup of $SO(4)$. The remaining elements $(p, r)^* = (-p, -r)^*$ are in the coset space[3] H_4/H_4'. We will explicitly show that the 7200 elements (p, r) of H_4' partition into 25 conjugacy classes and the remaining 7200 elements $(p, r)^*$ form an additional nine conjugacy classes, thereby totalling 34 conjugacy classes altogether.

Denote by (a, b) and $(a, b)^*$ arbitrary elements of H_4. Using (21) it is straightforward to show the following relations for conjugacy classes:

$$\begin{aligned}(a, b)(p, r)(a, b)^{-1} &= (ap\bar{a}, br\bar{b})\\ (a, b)^*(p, r)(a, b)^{*^{-1}} &= (a\bar{r}\bar{a}, \bar{b}\bar{p}b).\end{aligned} \tag{26}$$

This proves that the conjugacy classes of $2A_5$ in table 1 play an essential role, and moreover $(\bar{r}, \bar{p})$ belongs to the same conjugacy class of (p, r). Regarding the group elements $(p, r)^*$ the following relations are useful:

$$\begin{aligned}(a, b)(p, r)^*(a, b)^{-1} &= (apb, arb)^*\\ (a, b)^*(p, r)^*(a, b)^{*^{-1}} &= (a\bar{r}b, a\bar{p}b)^*.\end{aligned} \tag{27}$$

We note that p and $\bar{p}$ belong to the same conjugacy class of $2A_5$. We give the list of the classes in table 3 according to the orders of elements and including the total number of elements.

The class structures of the group elements $(p, r)^*$ can be worked out as follows. Denote by T the pair of elements $T = (p, r)^*$. Obviously $T^2 = (p\bar{r}, \bar{p}r)$ belongs to the set of elements of H_4'. Since elements with different orders belong to different conjugacy classes, it is better to classify the elements with respect to their orders. Let us assume that $T^{2m} = (1, 1) = (-1, -1)$ where m is an integer. This leads to the result $(p\bar{r})^m = (\bar{r}p)^m = \pm 1$ with possible values $m = 1, 2, 3, 4, 5, 6, 10$. But we note that $((p\bar{r})^m, (\bar{r}p)^m) = (1, 1) = ((p\bar{r})^m, (\bar{r}p)^m) = (-1, -1)$. This implies that one can restrict the values of m to $m = 1, 2, 3, 5$. We discuss each case separately.

(i) $m = 1$

$$\begin{aligned}T^2 &= I\\ p\bar{r} &= \bar{r}p = \pm 1.\end{aligned} \tag{28}$$

There are two solutions to (28):

[3] Coxeter and du Val use different notations for the finite subgroups of $O(4)$.

Table 3. Classes of the elements of type (p, r).

Class	Order	Type	# of elements
1	1	$(1, 1) = (-1, -1)$	1
2	2	$(-1, 1) = (1, -1)$	1
3	10	$(12_+, 1) \oplus (1, 12_+)$	24
4	5	$(12_-, 1) \oplus (1, 12_-)$	24
5	10	$(12'_+, 1) \oplus (1, 12'_+)$	24
6	5	$(12'_-, 1) \oplus (1, 12'_-)$	24
7	6	$(20_+, 1) \oplus (1, 20_+)$	40
8	3	$(20_-, 1) \oplus (1, 20_-)$	40
9	4	$(30, 1) \oplus (1, 30)$	60
10	5	$(12_+, 12_+) = (12_-, 12_-)$	144
11	10	$(12_+, 12_-) = (12_-, 12_+)$	144
12	5	$(12'_+, 12'_+) = (12'_-, 12'_-)$	144
13	10	$(12'_+, 12'_-) = (12'_-, 12'_+)$	144
14	5	$(12_+, 12'_+) \oplus (12'_+, 12_+)$	288
15	10	$(12_+, 12'_-) \oplus (12'_-, 12_+)$	288
16	15	$(12_+, 20_+) \oplus (20_+, 12_+)$	480
17	30	$(12_+, 20_-) \oplus (20_-, 12_+)$	480
18	15	$(12'_+, 20_+) \oplus (20_+, 12'_+)$	480
19	30	$(12'_+, 20_-) \oplus (20_-, 12'_+)$	480 ← Coxeter element
20	20	$(12_+, 30) \oplus (30, 12_+)$	720
21	20	$(12'_+, 30) \oplus (30, 12'_+)$	720
22	3	$(20_+, 20_+) = (20_-, 20_-)$	400
23	6	$(20_+, 20_-) = (20_-, 20_+)$	400
24	12	$(20_+, 30) \oplus (30, 20_+)$	1200
25	2	$(15_+, 15_+) \oplus (15_+, 15_-)$	450
		Total # of elements	7200

(a) $p = r$, $T = (p, p)^*, T^2 = I$. We have only 60 group elements of this type $(60_+, 60_+) = (60_-, 60_-)$

(b) $p = -r$, $T' = (p, -p)^*, T'^2 = I, T' = -T$ and the number of elements is 60 which can be written $(60_+, 60_-)$. They are obviously not in the same conjugacy class because $\operatorname{Tr} T' = -\operatorname{Tr} T$.

(ii) $m = 2$

$$T^4 = I \qquad (p\bar{r})^2 = (\bar{r}p)^2 = \pm 1. \tag{29}$$

$(p\bar{r})^2 = (\bar{r}p)^2 = 1$ is already covered in (i). Now we discuss the case $(p\bar{r})^2 = (\bar{r}p)^2 = -1$ which shows that they are pure quaternions. Let Q with ($Q_+ \in 15_+$ and $Q_- \in 15_-$) be pure quaternions and let $p\bar{r} = Q$. So for each value of r we have corresponding elements $p = Qr$. Possible choices are

$$p = \begin{cases} Q_+r_+ = Q_-r_- & 15 \times 60 = 900 \text{ elements} \\ Q_+r_- = Q_-r_+ & 15 \times 60 = 900 \text{ elements.} \end{cases}$$

Since $p\bar{r}$ and $\bar{r}p$ are in the same conjugacy class of $2A_5$ $\quad p = rQ$ does not lead to any other solution. Therefore the $T = (p, r)^*$ with $T^4 = I$ form a conjugacy class with 1800 elements:

(iii) $m = 3$

$$T^6 = I \qquad (p\bar{r})^3 = (\bar{r}p)^3 = \pm 1. \tag{30}$$

Table 4. Conjugacy classes of the elements $(p, r)^*$.

Class #	Order	Type	# of elements
26	2	$(p, p)^*$	60
27	2	$-(p, p)^*$	60
28	4	$(Qr, r)^*$, $Q^2 = -1$	1800
29	6	$(20_-r_+, r_+)^*$	1200
30	6	$-(20_-r_+, r_+)^*$	1200
31	10	$(12_+r_+, r_+)^*$	7200
32	10	$-(12_+r_+, r_+)^*$	7200
33	10	$(12'_+r_+, r_+)^*$	7200
34	10	$-(12'_+r_+, r_+)^*$	7200

Table 5. Characters of the classes (19) and (26).

	Conjugacy classes	
Characters	(19)	(26)
χ_4	$-\sigma$	-2
χ'_4	$-\tau$	-2
χ''_4	$-\sigma$	2
χ'''_4	$-\tau$	2

These are the elements in the class 20_-. Symbolically, we write $p = 20_-r$. Here again we have two distinct cases

$$T^6 = I \qquad p = 20_-r_+ : \qquad 20 \times 60 = 1200 \text{ elements}$$
$$T'^6 = I \qquad p = 20_-r_- : \qquad 20 \times 60 = 1200 \text{ elements}$$
$$T' = (p, r_-) = (p, r_+) = -T$$
$$\mathrm{Tr}\, T' = -\,\mathrm{Tr}\, T.$$

Therefore we have two conjugacy classes of elements with order 6 whose characters differ by a $(-)$ sign:

(iv) $m = 5$

$$T^{10} = I \qquad (p\bar{r})^5 = (\bar{r}p)^5 = \pm 1.$$

When we check table 1 we notice that there are four possibilities: 12_+, 12_-, $12'_+$, $12'_-$. Similar analysis leads to four more types of classes:

(1) $p = 12_+r_+ = 12_-r_-$ with $12 \times 60 = 720$ elements $T_+ = (p, r)^*$
(2) $p = 12_+r_- = 12_-r_+$ with $12 \times 60 = 720$ elements $T_- = -T_+$
(3) $p = 12'_+r_+ = 12'_-r_-$ with $12 \times 60 = 720$ elements $T'_+ = (p, r)^*$
(4) $p = 12'_+r_- = 12'_-r_+$ with $12 \times 60 = 720$ elements $T'_- = -T'_+$.

The conjugacy classes of the elements $(p, r)^*$ are listed in table 4.

The irreducible representations and their characters of any group can be determined when the generating relations of the group are given. We do not want to give the whole character table of H_4. The noncrystallographic Coxeter group H_4 has the following irreducible representations:

$$\mathbf{1}, \mathbf{1'}, \mathbf{4}, \mathbf{4'}, \mathbf{4''}, \mathbf{4'''}, \mathbf{6}, \mathbf{6'}, \mathbf{8}, \mathbf{8'}, \mathbf{9}, \mathbf{9'}, \mathbf{9''}, \mathbf{9'''}, \mathbf{10}, \mathbf{16}, \mathbf{16'}, \mathbf{16''}, \mathbf{16'''},$$
$$\mathbf{16^4}, \mathbf{16^5}, \mathbf{18}, \mathbf{24}, \mathbf{24'}, \mathbf{24''}, \mathbf{24'''}, \mathbf{25}, \mathbf{25'}, \mathbf{30}, \mathbf{30'}, \mathbf{36}, \mathbf{36'}, \mathbf{40}, \mathbf{48}.$$

Table 6. The Characters of the irreducible representations **4** and **4**$'$ of H_4.

Order	1	2	2	3	3	4	5	5	5
Elements	1	1	450	40	400	60	24	24	144
Class	$(1, 1)$	$(1, -1)$	(Q, Q)	$(20_-, 1)$	$(20, 20)$	$(30, 1)$	$(12'_-, 1)$	$(12_-, 1)$	$(12'_+, 12'_+)$
χ_4	4	-4	0	-2	1	0	-2σ	-2τ	σ^2
$\chi_{4'}$	4	-4	0	-2	1	0	-2τ	-2σ	σ^2
Order	5	5	6	6	10	10	10	10	
Elements	144	288	40	400	24	24	144	144	
Class	$(12_+, 12_+)$	$(12_+, 12'_+)$	$(20_+, 1)$	$(20_+, 20_-)$	$(12'_+, 1)$	$(12_+, 1)$	$(12'_+, 12'_-)$	$(12_+, 12_-)$	
χ_4	τ^2	-1	2	-1	2σ	2τ	$-\sigma^2$	$-\tau^2$	
$\chi_{4'}$	τ^2	-1	2	-1	2τ	2σ	$-\tau^2$	$-\sigma^2$	
Order	10	12	15	15	20	20	30	30	
Elements	288	1200	480	480	720	720	480	480	
Class	$(12_+, 12'_-)$	$(20_+, 30)$	$(12'_+, 20_+)$	$(12_+, 20_+)$	$(12'_+, 30)$	$(12_+, 30)$	$(12'_+, 20_-)$	$(12_+, 20_-)$	
χ_4	1	0	σ	τ	0	0	$-\sigma$	$-\tau$	
$\chi_{4'}$	1	0	τ	σ	0	0	$-\tau$	$-\sigma$	
Order	2	2	4	6	6	10	10	10	10
Elements	60	60	1800	1200	1200	7200	7200	7200	7200
Class	$(p, p)^*$	$-(p, p)^*$	(Qr, r)	$(20_+r_+, r_+)$	$(20_-r_-, r_+)$	$(12'_+r_-, r_+)$	$(12'_+r_+, r_+)$	$(12_+r_-, r_+)$	$(12_+r_+, r_+)$
χ_4	-2	2	0	1	-1	σ	$-\sigma$	τ	$-\tau$
$\chi_{4'}$	-2	2	0	1	-1	τ	$-\tau$	σ	$-\sigma$

There does not exist any standard notation in the literature to distinguish the irreducible representations of the same dimensionality. Our concern here of course is the branching of the eight-dimensional representation of $W(E_8)$ in terms of the irreducible representations of H_4. As we have already discussed $\mathbf{8} = \mathbf{4} + \mathbf{4}'$. To distinguish these four four-dimensional representations we picked up two characteristic conjugacy classes: the Coxeter element class # (19) and $(p, p)^*$ of class # (26). Their character values distinguish these four irreducible representations. In fact there is a simple relation between the characters of **4** and $\mathbf{4}'$: $\chi'_4 = \chi_4(\sigma \to \tau)$. The characters of these two irreducible representations are given in table 6.

7. Concluding remarks

The noncrystallographic symmetries with five-fold symmetry in two, three and four dimensions are best described by icosians (when embedding them in crystallographic groups $W(A_4)$, $W(D_6)$ and $W(E_8)$ in respective four-, six- and eight-dimensional spaces). By using the reflection generators of $W(E_8)$ we have transformed the generators of H_4 into block-diagonal form. We have constructed the Coxeter element of $W(E_8)$ as well as of H_4 and have shown that the characteristic polynomial of the Coxeter element of $W(E_8)$ can be written as the product of two polynomials, one corresponding to the characteristic polynomial of the Coxeter element of H_4. By using icosians we have partitioned H_4 into its conjugacy classes and determined the characters of two four-dimensional irreducible representations of H_4.

It is obvious that the noncrystallographic H_3 and its embedding into the crystallographic group $W(D_6)$ in six-dimensional space are very useful for the quasicrystals of icosahedral symmetry. We are optimistic that the embedding of H_4 in E_8 will also shed light on the studies of the heterotic string theory.

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Appendix

$W(E_8)$ generators in the basis of $e_a, \sigma e_a (a = 0, 1, 2, 3)$

$$r_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad r_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad r_2' = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$r_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad r_3' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_4 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad r_4' = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

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